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Geometrical frustration in 2D optical patterns

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Abstract. In the case of 2D optical patterns, frustration comes from the interplay between the physical constraints (light-matter interaction) and the geometrical constraints (cavity length and structure). Depending on the dynamical parameters, we are able to single out two distinct behaviors. For small diffusion and close to threshold, the system is forced to fulfill the geometrical constraints giving rise to a phase dynamics of quasicrystals. For larger diffusion, the system fragmentates into spatial domains giving rise to a competition between different patterns. By means of a geometrical argument, we show that the spatial distribution of domains is related to the symmetry imposed by the geometrical constraint and that the domain borders are disinclination defects. These defects being the nucleation centers of spatial domains, they trigger the onset of pattern competition.

PACS. 42.60.Jf Beam characteristics: profile, intensity, and power; spatial pattern formation – 42.79.Kr Display devices, liquid-crystal devices – 42.65.Pc Optical bistability, multistability, and switching

1 Introduction

In the Euclidean space, it is well-known that only the rotations of order 2, 3, 4, and 6 are compatible with translations in order to obtain a perfect tessellation. Indeed, a tessellation is a perfect tiling with only one elementary pattern for a given space with given curvature and dimension. Examples of tessellations in a two-dimensional space can be find in Escher's paintings [1].

Geometrical frustration [2] arises from the impossibility to tessellate the Euclidean space with a $N > 4$ order rotation patterns. For this types of rotation order patterns, disinclination defects appear. Indeed, a disinclination defect is associated to the breaking of a rotation symmetry, as well as a dislocation is associated to the breaking of a translation symmetry. A disinclination defect may be generated by cutting the structure along a line and by adding (or subtracting) a part of the matter between the two cuts. For example, it is not possible to make a perfect tiling, *i.e.* a tessellation, of the plane with regular pentagons because of the pentagon inner angles. Indeed, if five pentagons are surrounding a central one, there will be spacements (cuts) between two neighboring pentagons. These spacements are disinclination defects. The border of the cuts are equivalent under the effect of the N-order rotation related to the symmetry of the underlying pattern. So, in our example, there are five cuts surrounding the central pentagon, and these five cuts are equivalent under a $2\pi/5$ rotation symmetry.

Geometrical frustration appears in quasicrystals, as well as in several classes of crystals like the Frank-Kasper phases [3,4], and in complex fluids like the cholesteric blue phases [5]. Optical pattern formation has been demonstrated as a fruitful area for the study of nonlinear phenomena and symmetry selection problems, showing strong analogies with other areas as fluid dynamics, chemistry and biology [6]. In particular, optics provides easily two-dimensional patterns which arise from amplitude modulation of the transverse profile of an optical beam as it passes through a nonlinear medium [7]. Roll-hexagon transition [8], crystals and quasicrystals [9] and domain competition of different patterns [10] have been reported for a system based on a Liquid-Crystal-Light-Valve (LCLV) with a nonlocal feedback [11].

We want to discuss here the problem of geometrical frustration that can arise in this system when physical and geometrical constraints are not commensurate. For example, the feedback loop can impose a N-fold overall symmetry which is incommensurate with the 3-fold symmetry naturally selected by the physical interactions. We think that our considerations are not related to this specific system but that they can apply to any generic 2D pattern forming system. The advantage of the experiment LCLV with feedback, is that we can easily adjust and control the degree of geometrical frustration we want to impose to the system.

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We shall present the experimental setup and the theory of the LCLV in Section 2. In Section 3, the 2D quasicrystals, as well as the pattern competition, are shown and discussed. We show that both the behaviors are a manifestation of geometrical frustration. And Section 4 is the conclusion.

2 Experiments and LCLV theory

A LCLV is a essentially a mirror sandwiched between a nematic liquid crystal layer and a photoconductive layer. An AC voltage is applied between the photoconductor and the liquid crystal layer. In the presence of light on the photoconductor, the voltage drop across the liquid crystal layer increases inducing a reorientation of the LC molecules. For the reflected light, this reorientation produces a phase change. Transverse patterns in the optical field are due to diffraction which converts the phase modulation, generated within the LCLV, into amplitude modulation. This latter one, on its turn, modifies the properties of the LCLV. In this way, a positive feedback is realized for all those spatial frequencies satisfying the resonance conditions $q^2L/2k = \pi/2, 3\pi/2, ...$, where q is the field transverse wavenumber, $k = 2\pi/\lambda$ is the optical wavenumber and L is the free propagation length. In our case $\lambda = 632$ nm is the wavelength of an He–Ne laser and L can vary between 20 and 60 cm. The laser beam is expanded up to a diameter of 2.5 cm which is the transverse size of the LCLV. By inserting a circular diaphragm of 1 cm diameter in front of the LCLV, only a central region is let to be active so that the medium is uniformly illuminated. Indeed, the intensity profile of the laser beam can be considered flat on its central region and it is not playing any role in the stability of the observed patterns.

The natural symmetry of the transverse patterns here arising is the hexagonal one, due to the quadratic character of the light-matter interaction [12]. The fundamental pattern size is the one satisfying the resonance condition, that is

$$q^{-1} \simeq (\lambda L)^{-1/2}, \tag{1}$$

that is, of the order of a few tenth of mm for the parameters currently set in the experiments.

The feedback is realized by means of a coherent optical fiber bundle. By twisting the bundle, the image on the back of the LCLV can be rotated by any angle $\varDelta = 2\pi/N$ with respect to the front image. This rotation impose an overall N-fold symmetry which can or not be consistent with the natural hexagonal symmetry provided by the medium nonlinearity [8–12]. The geometrical frustration comes from the interplay between the N-fold symmetry imposed by the feedback rotation (boundary conditions) and the 3-fold symmetry imposed by the light-matter interaction (physical interaction).

Moreover, the feedback rotation angle permits to destabilize all the successive bands provided by diffraction independently of the sign of the nonlinearity, *i.e.* of whether we use a focusing or defocusing medium [9]. Normalizing to the basic wavenumber $q_0 = \sqrt{2}\pi/\sqrt{\lambda L}$, all

$$q_j = \sqrt{2j+1} \tag{2}$$

for $j = 0, 1, 2, 3, ...$ can be destabilized. In the real space, this defines a characteristic length $\ell_j = 2\pi/q_j$. In the Fourier space (far field), this allows us to define Fourier circles of radius q_j. Excited modes are wavevectors lying on these circles.

Actually, a frequency cutoff is imposed by the diffusion length $l_{\rm D}$ intrinsic to the nonlinear medium. The marginal stability curve *versus* q^2 is made of a series of bands (instability balloons) whose minima lie on a line of slope $\sigma^{-1} = l_{\rm D}^2/\lambda L$. More precisely, the marginal stability curve is composed by two branches

$$I_{\rm th} = \frac{(1+\theta/\sigma)}{(\sin\theta)} \qquad \text{for } N \text{ even}, \tag{3}$$

$$I_{\rm th} = \frac{(1+\theta/\sigma)}{(\cos(\pi/N)\sin\theta)} \qquad \text{for } N \text{ odd}, \tag{4}$$

and

$$I_{\rm th} = -\frac{(1+\theta/\sigma)}{(\sin\theta)} \qquad \text{for } N \text{ even or odd}, \tag{5}$$

where $\theta = \sigma q^2$. The positive branches of these two curves give the threshold value $I_{\rm th}$ of the input intensity required to excite a mode with a wavenumber q. The positive branches of this curve are represented in Figure 1 for $N = 5$ and 7 and for the two cases of small diffusion ($\sigma = 300$) and large diffusion ($\sigma = 30$).

Here, σ is a nondimensional parameter measuring the strength of diffraction with respect to diffusion. In our setup σ can be changed by varying the free propagation length L. For large σ successive minima are almost aligned so that many unstable bands can be simultaneously excited as the input intensity goes slightly above the threshold of the first band. In this situation, the observed patterns can be quasicrystals or superlattices, depending on whether or not geometrical frustration comes into play.

Superlattices [14] are complex crystals composed by wavevectors lying on different Fourier circles, at variance with simple crystals where couplings are established only between wavevectors lying on the same Fourier circle. In the LCLV experiment, superlattices are observed if N satisfies the natural hexagonal symmetry ($N = 0, 3, 6, ...$), since in this case there is no geometrical frustration [13]. Superlattices are observed also for $N = 2, 4, 8, ...$ where the imposed 4-fold symmetry overcomes the hexagonal one producing metastable superlattices of order 4. In this case, as in the 3-fold case, there is no geometrical frustration since the quadratic couplings (giving rise to hexagons when realized on a single circle) can be established over different Fourier circles.

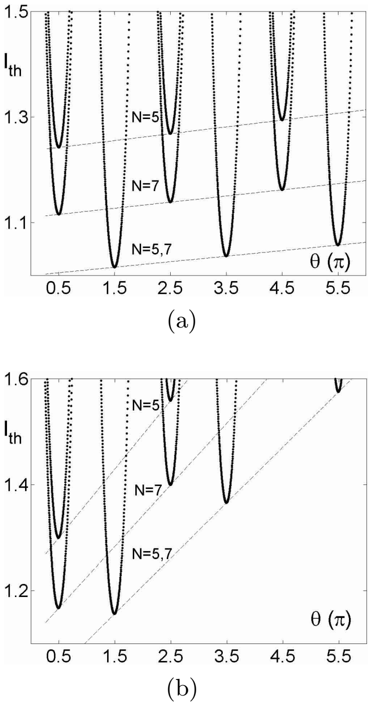

Fig. 1. Marginal stability curve for $N = 5, 7$ and for (a) $\sigma = 300$ and (b) $\sigma = 30$.

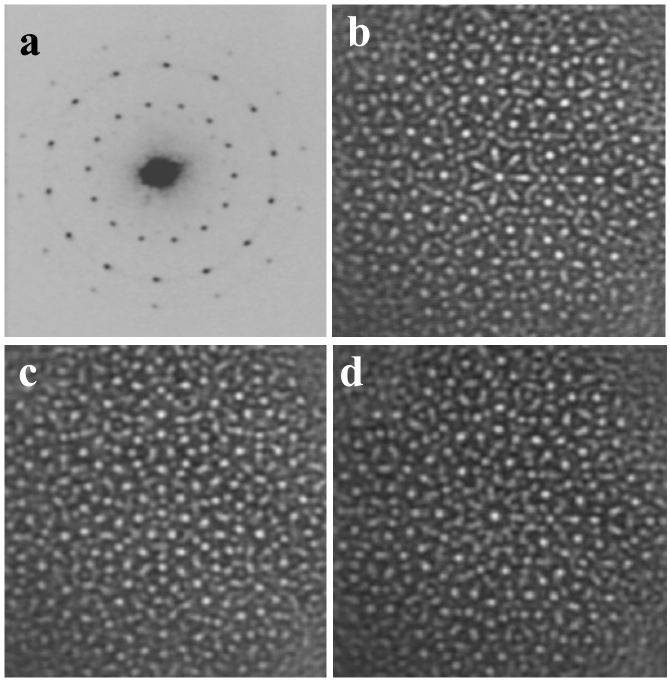

Fig. 2. Quasicrystalline patterns observed in the LCLV experiment for $N = 7$. The same far-field shown in (a) corresponds to different near fields as shown in (b), (c) and (d). These images correspond to phason modes, *i.e.*, different patterns realized with different phase relationships between the same wavevectors.

3 Geometrical frustration in LCLV patterns: results and discussion

As long as N is odd and different from 3, we have geometrical frustration. In this case even if many Fourier circles can be simultaneously excited, it is not possible to fulfill the geometrical constraint by realizing quadratic couplings between wavevectors lying on different circles. Moreover, when the system is close to the threshold, the wavevectors lengths are close to the values corresponding to the minima of the marginal stability curve. Then, the patterns are quasicrystals composed by wavevectors lying on successive Fourier circles with incommensurate radii [9]. In this case, the observed patterns have an N-fold rotational order and a quasiperiodic instead of periodic translational order.

The geometrical frustration manifests itself as a phase dynamics characterized by a few modes alternating in time. These modes are global phasons corresponding to spatial configurations realized with a given phase difference between wavevectors lying on the excited Fourier circles. Different phasons correspond to different images in the near field whereas the far-field (Fourier circles) remains unchanged. We show in Figure 2 an example of phasons in the case of $N = 7$.

When diffusion increases (low σ), only the first two branches of the marginal stability curve have almost aligned minima (see Fig. 1b). In this situation, and close to threshold, only two Fourier circles can be simultaneously excited, corresponding to the minimum of the first N-dependent branch and to the minimum of the first N-independent branch. The patterns associated with these two minima have, respectively, a characteristic length scale $\ell_0 = 2\pi/q_0$ and $\ell_1 = 2\pi/q_1$.

Moreover, the two patterns are qualitatively different. Indeed, to the N-dependent branches is associated an eigenvalue with a non-zero imaginary part which adds a rotation in time of the selected wavevector [9]. The corresponding pattern is composed of rolls whose orientation rotates in the course of time. On the contrary, the eigenvalue associated to the N-independent branches is purely real so that the corresponding pattern is composed of steady hexagons [9,12].

The excitation of successive Fourier circles becoming energetically too expensive, geometrical frustration is now expressed in a different way. Instead of realizing quasicrystalline configurations, the system alternates between the two first unstable bands, giving rise to a spatio-temporal competition between the two associated patterns, that is, between hexagons and rolls [10]. Experimental snapshots showing the simultaneous presence of the two patterns onto different spatial domains are reported in Figures 3 and 4 for $N = 5$ and 7 respectively.

A qualitative explanation of this behavior can be given on the basis of geometrical considerations. Indeed, because of geometrical frustration, as one pattern bifurcates from threshold, it presents defects. These defects are centers of nucleation for the other pattern, so that spatial domains of rolls grow inside hexagons and vice versa. To account for domain distribution, we have to fill the active area

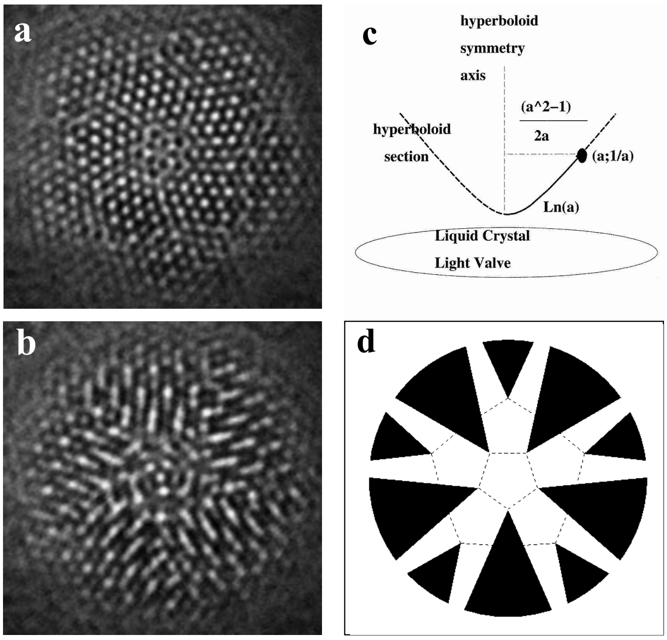

Fig. 3. Domain formation for $N = 5$ (experimental images). Disinclination lines arise in the (a) hexagons and (b) rolls patterns. We show in (c) a section of a 2D hyperboloid H^2. For a point located in coordinates $(a; 1/a)$, we report its distance from the H^2 symmetry axis (dot-dashed line) together with its distance on H^2 (solid line on the H^2 section). We show in (d) H^2 unfolded on the plane by respecting a 5-fold symmetry. Black areas represent the disinclinations which arise from the five cuts that we need to operate in order to eliminate the non-zero curvature of H^2.

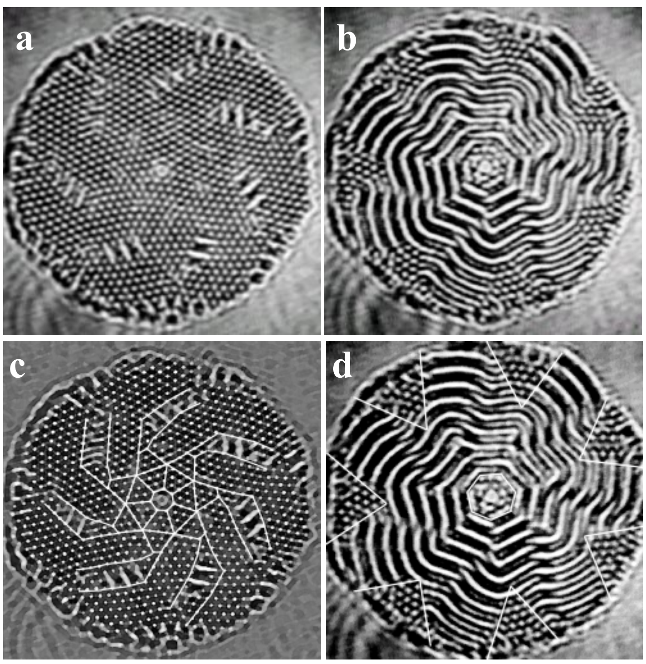

Fig. 4. Domain competition for $N = 7$ (experimental images). (a, b) Rolls and hexagons exchange their stability nucleating from the disinclination defects. (c, d) Same images with disinclination borders marked by white lines.

of the LCLV by combining two symmetries of rotation: the 3-fold symmetry, which is naturally selected by the quadratic nature of the light-matter interaction, and the N-fold symmetry which is imposed by the rotation in the feedback loop. By selecting $N = 5$, or $N = 7$, we are dealing with two incommensurate symmetries.

In other words, geometrical frustration comes from the fact that we have to fit hexagons on a 5-fold, or 7-fold symmetry. As long as hexagons are a tessellation of the plane in the Euclidean space, it is not possible to tessellate the same plane with a 5-fold (resp. 7-fold) symmetry. This tessellation becomes possible on a 2D hyperboloid, H^2. We will define here H^2 as a two dimensional hyperboloid plunged in the 3D real space. So the definition of the H^2 plane corresponds here to a surface obtained by rotating a 1D hyperbole around its symmetry axis (see Fig. 3c).

Once the tessellation has been made, one has to unfold H^2 on the plane in order to recover the geometrical features of the spatial domains in the LCLV patterns. Indeed, H^2 is built in the 3D real space and has a constant negative curvature. As we show in Figure 3c, if we use the metric of the 3D real space R^3 on H^2, a point of Cartesian coordinates $(a; 1/a)$ (black point) will be located on a circle of radius $(a^2 - 1)/2a$ (dot-dashed line), centered on the symmetry axis of H^2 (dashed line). Thus the perimeter of this circle is equal to $2\pi(a^2 - 1)/2a$. When unfolding H^2, the point located on $(a; 1/a)$ will be at a distance $(\ln a)$ of the top of H^2 (this distance is represented by a solid line). Thus on a circle of perimeter $2\pi \ln a$. Cuts will appear in the unfolded H^2 as soon as $\ln a - (a^2 - 1)/2a > \ell_0$ or ℓ_1 depending on whether the basic pattern is prevalently composed by hexagons (basic length ℓ_0) or rolls (basic length ℓ_1).

A rough approximation of the unfolding of H^2 is the crash of an empty egg shell: if we apply an uniaxial pressure on an half empty egg shell, this one will break into pieces in order to adapt its curvature to the curvature of a plane. The same phenomenon will happen for our hyperboloid: the underlying patterns (hexagons or rolls) will keep their geometry but there will appear cuts between them. In our case, these cuts will respect the underlying N-fold symmetry, as we show in Figure 3d for $N = 5$. One has to add that the cuts are not necessarily straight from the center of the hyperboloid, as one may see in the following.

Let us define R as the distance from one point of H^2 to the center of the hyperboloid. As an example, our point located in $(a; 1/a)$ is at a distance $R = \ln a$. Hence, a disinclination defect will appear as soon as the distance $R - \ell_j$ ($j = 1$ for hexagons and $j = 0$ for rolls) is equal to $(\exp(2R) - 1)/2\exp(R)$. Indeed, when unfolding H^2, cuts appear in this unfolded space, because unfolding corresponds to a transformation from a non zero curvature to a zero one. The inside of the cuts may then be filled with patterns which corresponds to the physical characteristics of the LCLV (hexagons or rolls). It is clear , as in all disinclinations defects, that the boundaries of the cuts respect the order of symmetry of the H^2 tessellation [2].

Now, we can see on the figures that hexagons, or rolls, are distributed over spatial domains which present a 5-fold (7-fold) symmetry. These domains can be thought of as the elementary pentagons (heptagons) which tessellate H^2. The cuts between adjacent domains are the disinclinations defects coming from the unfolding procedure of H^2 over the plane.

Figure 3a corresponds to a low excitation energy of the marginal stability curve and to $N = 5$. Hexagons are here dominant. Leaving the center and going outwards, geometrical frustration appears: there are hexagons regularly tiling the inside of five large pentagons. When the cuts in the hyperboloid become larger than the size ℓ_0 of the rolls, disinclination defects appear, *i.e.* the borders of the previous five large pentagons are filled with rolls. The borders of the regions tiled with hexagons correspond to a 5-fold symmetry (presence of 5 branches). Besides ℓ_1, the characteristic length of hexagons, a larger length appears which is related to the 5-fold symmetry and which is the characteristic size of domains. Indeed, hexagons are distributed over five large domains separated by disinclination defects.

In Figure 3b, the input intensity is higher and ℓ_0, the characteristic length of rolls, is dominant. Here the pattern adapts itself to adjust at best a 5-fold tiling with rolls. The 3-fold symmetry is appearing in the outer region (presence of hexagons) and may once again associated with disinclination defects. Indeed, in the outer region the value of $R - (\exp(2R) - 1)/2\exp(R)$ is larger than the size ℓ_1 of the hexagons which are filling these disinclination defects.

We have to add that in both the cases of Figures 3a and 3b, in the center of the pattern there is a region composed of hexagons which are arranged in a 5-fold symmetry, even when the rolls are dominant. Indeed, this central region is not large enough to accommodate rolls in a 5-fold symmetry, due to the fact that the size of rolls is larger than the size over which the curvature of the plane and H^2 almost coincide. This explains why it is always the hexagonal solution, that is characterized by a smaller length, which fills the central region of the pattern.

Figure 4 correspond to the case $N = 7$. The input intensity is adjusted in such a way that there is a continuous competition between hexagons and rolls [10]. In Figure 4a (correspondingly c) ℓ_1, the length of hexagons, is dominant. In Figure 4b (correspondingly d) ℓ_0, the length of rolls, is dominant and it appears in seven large domains of rolls. In both cases, that is, hexagons or rolls dominant, the antagonistic solution is filling the disinclination defects. In Figures 4c and 4d, we show a processed version of the experimental snapshots, where the borders of the disinclination defects are marked by white lines. As said before, the cuts are not necessarily straight.

The competition between ℓ_0 and ℓ_1 comes from a continuous nucleation of one pattern into the other. The nucleation centers are the disinclination defects generated by the geometrical frustration. The images correspond to a tiling which is the closest to a regular tiling of the plane on which a 7-fold symmetry has been applied. Spatial domains reflects the 7-fold symmetry which has been imposed by the rotation in the feedback loop. Disinclination defects are the cuts resulting from the matching of the unfolded H^2 with the plane.

As a summary, the tiling of Figure 3 (4) corresponds to the 5-fold (7-fold) symmetry in the center of the LCLV active area. There is a singular region in the center of the unfolded H^2 where it is not possible to make a tessellation with rolls (due to their size). So, even when the rolls are dominant, the central region of the images is filled with hexagons adjusting a 5-fold (resp7-fold) symmetry. As soon as we move away from the center, disinclination defects appear which are filled respectively with rolls or hexagons, at variance with the surrounding pattern. The domain boundaries correspond to the borders of the cuts of the unfolded H^2. As a whole, the 5 (7-fold) symmetry is respected and conserved during the pattern competition. Spatial domains grow from disinclination defects and they are tiled with rolls or hexagons, with a continuous exchange between the two patterns.

4 Conclusion

The LCLV experiment allows to obtain geometrically frustrated optical patterns. We have shown that, depending on the dynamical parameters, geometrical frustration gives rise to two distinct behaviors.

For large σ (small diffusion) many elementary wavenumbers can be simultaneously excited, so that patterns are two-dimensional quasicrystals. In this case, geometrical frustration manifests itself as a phase dynamics. An alternation between a few phason modes is observed.

For small σ (large diffusion), and close to threshold, only two wavenumbers are excited. The corresponding modes are rotating rolls and steady hexagons. Geometrical frustration gives rise to a fragmentation into spatial domains leading to a competition between rolls and hexagons. The spatial arrangement of domains reflects the N-fold symmetry imposed by the geometrical constraints (rotation in the optical feedback loop). By selecting N odd and different from 3, frustration comes from the combination of the N-fold symmetry with the 3-fold symmetry naturally selected by the light-matter interaction.

The matching between these two symmetries can be realized by unfolding on the plane an hyperboloid, H^2, over which N-gons, with $N > 4$, constitutes a tessellation. The matching cannot be perfect and leaves uncovered regions between the N cuts. These regions are disinclinations defects. Spatial domains of rolls (hexagons) into hexagons (rolls) grow from the disinclination defects, leading to a continuous competition between hexagons and rolls.

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